Computer Graphics III
Spherical integrals, Light & Radiometry – Exercises

Jaroslav Křivánek, MFF UK

Jaroslav.Krivanek@mff.cuni.cz

Surface area of a (subset of a) sphere

- Calculate the surface area of a unit sphere.
- Calculate the surface area of a spherical cap delimited by the angle θ_0 measured from the north pole.
- Calculate the surface area of a spherical wedge with angle ϕ_0.

Solid angle

- What is the solid angle under which we observe an (infinite) plane from a point outside of the plane?

- Calculate the solid angle under which we observe a sphere with radius R, the center of which is at the distance D from the observer.

Isotropic point light

- **Q:** What is the emitted power (flux) of an isotropic point light source with intensity that is a constant I in all directions?

Isotropic point light

- **A:** Total flux:

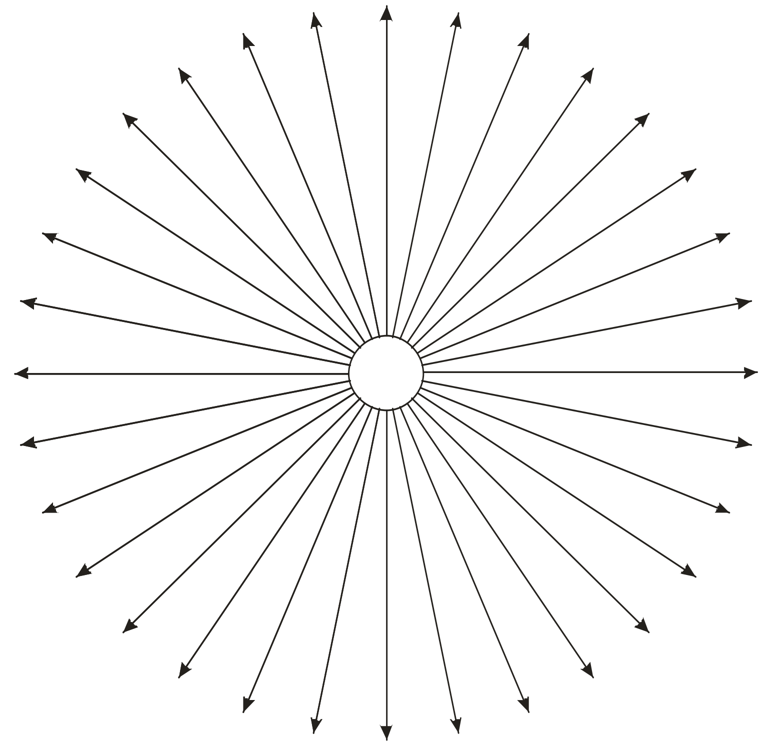
$$\Phi = \int_\Omega I(\omega)\,\mathrm{d}\omega = \left| \begin{array}{c} \textit{substitute}: \\ \mathrm{d}\omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \end{array} \right|$$

$$= I \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$$

$$= I\,2\pi \left[-\cos\theta\right]_0^\pi$$

$$= 4\pi I$$

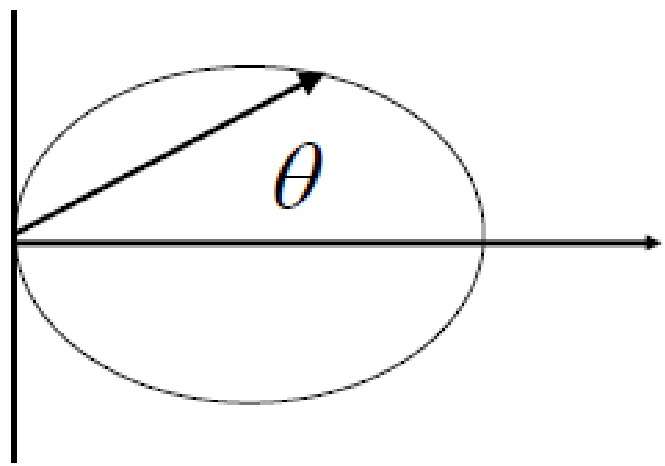
$$I = \frac{\Phi}{4\pi}$$



Cosine spot light

- What is the power (flux) of a point source with radiant intensity given by:

$$I(\omega) = I_0 \max\{0, \omega \cdot \vec{d}\}^s$$



Spotlight with linear angular fall-off

- What is the power (flux) of a point light source with radiant intensity given by:

$$I(\theta, \phi) = \begin{cases} I_0 & \theta \leq \alpha \\ I_0 \dfrac{\beta - \theta}{\beta - \alpha} & \alpha < \theta < \beta \\ 0 & \theta \geq \beta \end{cases}$$

Constant part

$$\Phi_1 = \int_0^{2\pi} \int_0^{\alpha} I_0 \sin\theta d\theta d\phi = I_0 2\pi(1-\cos\alpha).$$

Linear part

$$\Phi_2 = \int_0^{2\pi} \int_\alpha^{\beta} I_0 \frac{\beta-\theta}{\beta-\alpha} \sin\theta d\theta d\phi = I_0 \frac{2\pi}{\beta-\alpha} \int_\alpha^{\beta} (\beta-\theta) \sin\theta d\theta \tag{1}$$

The last integral is the sum of the following two integrals:

$$\int_\alpha^{\beta} \beta \sin\theta d\theta = \beta\cos\alpha - \beta\cos\beta \tag{2}$$

$$-\int_\alpha^{\beta} \theta \sin\theta d\theta = \Big|\sin\theta - \theta\cos\theta\Big|_\beta^\alpha = \sin\alpha - \alpha\cos\alpha - \sin\beta + \beta\cos\beta \tag{3}$$

Plugging (2) and (3) into (1) and rearranging, we get

$$\Phi_2 = I_0 \frac{2\pi}{\beta-\alpha}[(\beta-\alpha)\cos\alpha + \sin\alpha - \sin\beta] = I_0 2\pi \left[\cos\alpha - \frac{\sin\beta - \sin\alpha}{\beta-\alpha}\right]. \tag{4}$$

Total flux

$$\boxed{\Phi = \Phi_1 + \Phi_2 = I_0 2\pi \left[1 - \frac{\sin\beta - \sin\alpha}{\beta-\alpha}\right]} \tag{5}$$

Irradiance due to a Lambertian light source

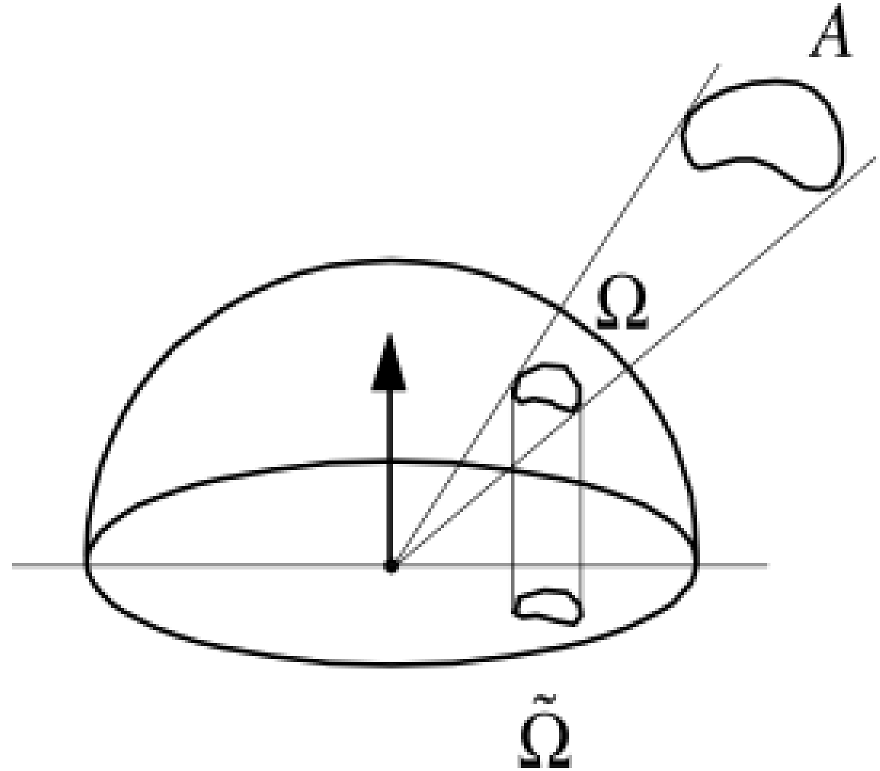
- What is the irradiance $E(\mathbf{x})$ at point $\mathbf{x}$ due to a uniform Lambertian area source observed from point $\mathbf{x}$ under the solid angle Ω?

Uniform Area Source

$$E(x) = \int_{H^2} L \cos\theta \, d\omega$$

$$= L \int_{\Omega} \cos\theta \, d\omega$$

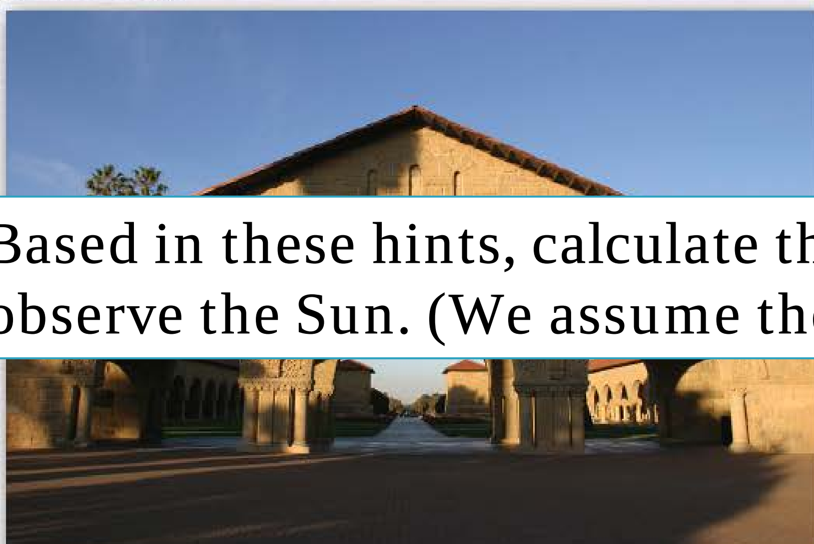
$$= L\tilde{\Omega}$$



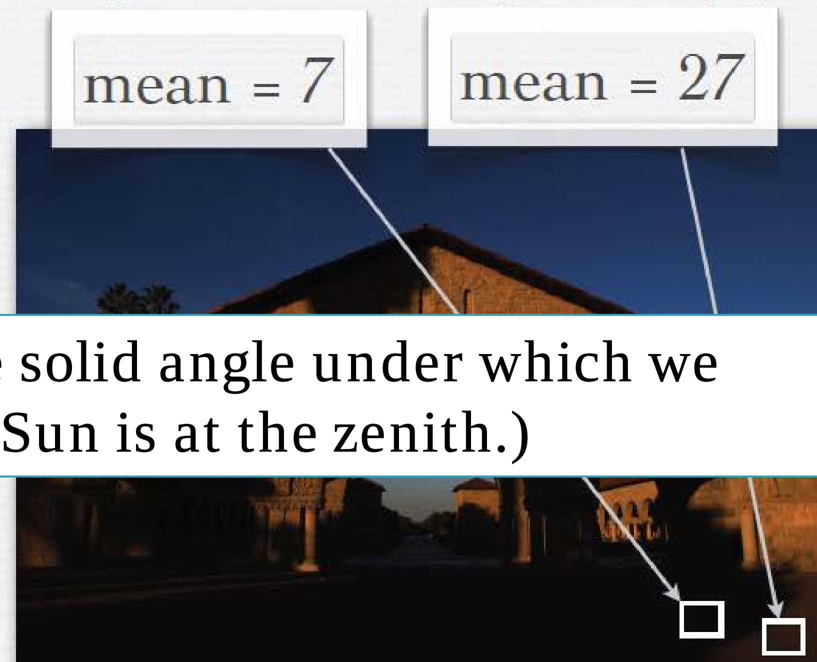
How dark are outdoor shadows?

- luminance arriving on a surface from a full (overhead) sun is 300,000 × luminance arriving from the blue sky, but the sun occupies only a small fraction of the sky
- illuminance on a sunny day = 80% from the sun + 20% from blue sky, so shadows are 1/5 as bright as lit areas (2.3 f/stops)

(Marc Levoy)



JPEG file



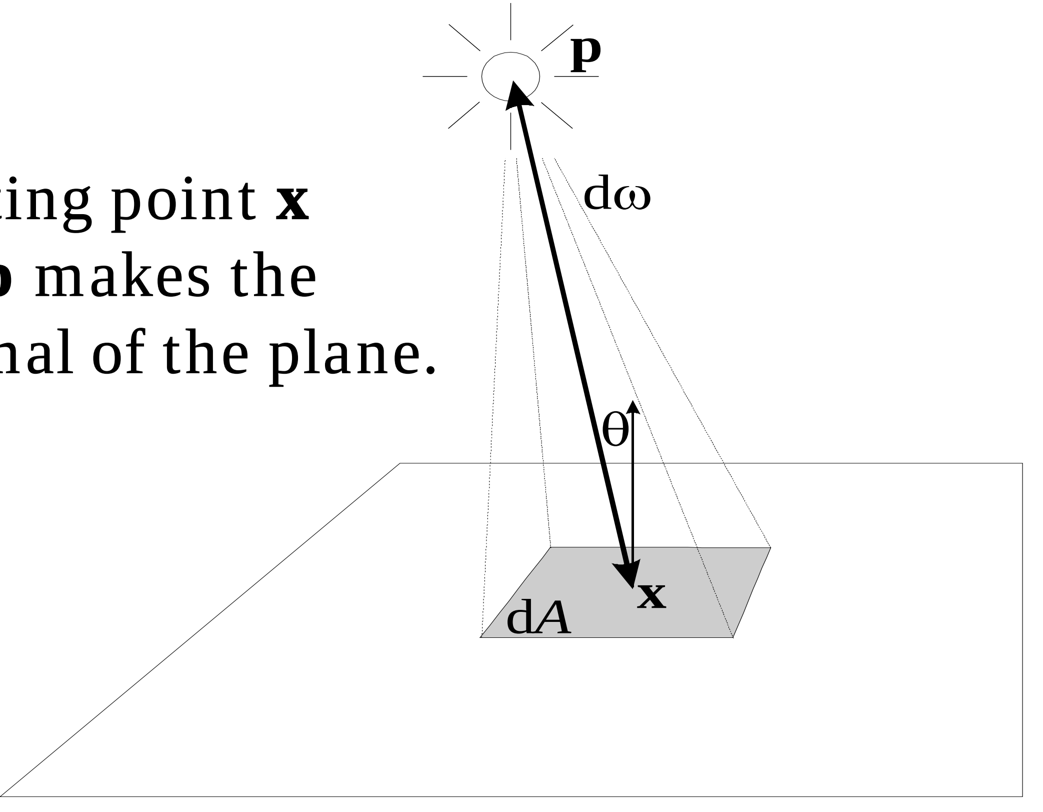
RAW, linearly boosted

Based in these hints, calculate the solid angle under which we observe the Sun. (We assume the Sun is at the zenith.)

Irradiance due to a point source

- What is the irradiance at point **x** on a plane due to a point source with intensity $I(\omega)$ placed at the height h above the plane.

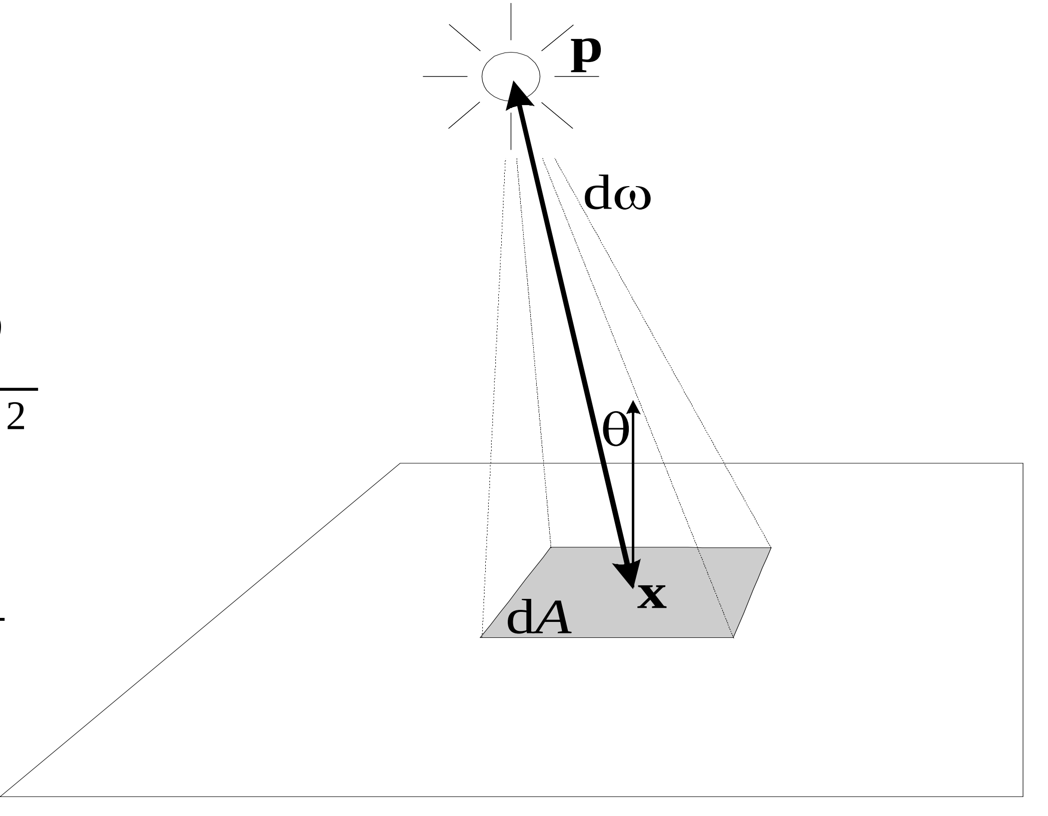
- The segment connecting point **x** to the light position **p** makes the angle θ with the normal of the plane.



Irradiance due to a point source

- Irradiance of a point on a plane lit by a point source:

$$
\begin{aligned}
E(\mathbf{x}) &= \frac{d\Phi(\mathbf{x})}{dA} \\
&= \frac{I(\mathbf{p} \to \mathbf{x})\, d\omega}{dA} \\
&= I(\mathbf{p} \to \mathbf{x}) \frac{\cos\theta}{\|\mathbf{p} - \mathbf{x}\|^2} \\
&= I(\mathbf{p} \to \mathbf{x}) \frac{\cos^3\theta}{h^2}
\end{aligned}
$$



Area light sources

- Emission of an area light source is fully described by the emitted radiance $L_e(\mathbf{x},\omega)$ for all positions on the source $\mathbf{x}$ and all directions ω.

- The total emitted power (flux) is given by an integral of $L_e(\mathbf{x},\omega)$ over the surface of the light source and all directions.

$$\boxed{\Phi = \int_A \int_{H(\mathbf{x})} L_e(\mathbf{x},\omega) \cos\theta \, \mathrm{d}\omega \, \mathrm{d}A}$$

Diffuse (Lambertian) light source

- What is the relationship between the emitted radiant exitance (radiosity) $B_e(\mathbf{x})$ and emitted radiance $L_e(\mathbf{x}, \omega)$ for a Lambertian area light source?

Lambertian source =

emitted radiance does not depend on the direction ω

$$L_e(\mathbf{x}, \omega) = L_e(\mathbf{x}).$$

Diffuse (Lambertian) light source

- $L_e(\mathbf{x}, \omega)$ is constant in ω
- Radiosity: $B_e(\mathbf{x}) = \pi L_e(\mathbf{x})$

$$
\begin{aligned}
B_e(\mathbf{x}) &= \int_{H(\mathbf{x})} L_e(\mathbf{x}, \omega) \cos\theta \,\mathrm{d}\omega \\
&= L_e(\mathbf{x}) \int_{H(\mathbf{x})} \cos\theta \,\mathrm{d}\omega \\
&= \pi \, L_e(\mathbf{x})
\end{aligned}
$$

Uniform Lambertian light source

- What is the total emitted power (flux) Φ of a **uniform** Lambertian area light source which emits radiance L_e
  - Uniform source – radiance does not depend on the position, $L_e(\mathbf{x}, \omega) = L_e = \text{const.}$

Uniform Lambertian light source

- $L_e(\mathbf{x}, \omega)$ is constant in $\mathbf{x}$ and $\boldsymbol{\omega}$

$$\boldsymbol{\Phi_e = A\, B_e = \pi\, A\, L_e}$$